PROPERTIES OF TRIANGLES

1. The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence is called circumcentre of the triangle. If S is the circumcentre of ΔABC, then SA = SB = SC. The circle with center S and radius SA passes through the three vertices A, B, C of the triangle. This circle is called circumcircle of the triangle. The radius of the circumcircle of ΔABC is called circumradius and it is denoted by R.

2. Sine Rule : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$

   $\therefore$ a = 2R sin A, b = 2R sin B, c = 2R sin C.

3. Cosine Rule : $a^2 = b^2 + c^2 - 2bc \cos A$, $b^2 = c^2 + a^2 - 2ca \cos B$, $c^2 = a^2 + b^2 - 2ab \cos C$.

4. $\cos A = \frac{b^2+c^2-a^2}{2bc}$, $\cos B = \frac{c^2+a^2-b^2}{2ca}$,

   $\cos C = \frac{a^2+b^2-c^2}{2ab}$.

5. Projection Rule : a = b cos C + c cos B, b = c cos A + a cos C, c = a cos B + b cos A.

6. Tangent Rule or Napier's Analogy : $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$,

   $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$,

   $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$.

7. Mollweide Rule :

   $$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}, \quad \frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$$

8. $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$.

9. $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$.

10. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$, $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

11. $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$,

$\tan\frac{B}{2} = \frac{\Delta}{s(s-b)} = \frac{(s-c)(s-a)}{\Delta}$,

$\tan\frac{C}{2} = \frac{\Delta}{s(s-c)} = \frac{(s-a)(s-b)}{\Delta}$.

12. $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}, \cot\frac{B}{2} = \frac{s(s-b)}{\Delta}, \cot\frac{C}{2} = \frac{s(s-c)}{\Delta}$

13. Area of ΔABC is $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}\sin C = 2R^2 \sin A \sin B \sin C =$

$\frac{abc}{4R}\sqrt{s(s-a)(s-b)(s-c)}$.

14. $r = \frac{\Delta}{s} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = \frac{a}{\cot\frac{B}{2}+\cot\frac{C}{2}} = \frac{b}{\cot\frac{C}{2}+\cot\frac{A}{2}}$

$= \frac{c}{\cot\frac{A}{2}+\cot\frac{B}{2}}$

15. $r_1 = \frac{\Delta}{s-a} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = s\tan\frac{A}{2} = (s-b)\cot\frac{C}{2} = (s-c)\cot\frac{B}{2} = \frac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}$.

16. $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = (s-c)\cot\frac{A}{2} = (s-a)\cot\frac{C}{2} =$

$4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b}{\tan\frac{C}{2}+\tan\frac{A}{2}}$.

17. $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = (s-a)\cot\frac{B}{2} = (s-b)\cot\frac{A}{2} =$

$\frac{c}{\tan\frac{A}{2}+\tan\frac{B}{2}}$.

18. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{1}{r}$.

19. $r\, r_1 r_2 r_3 = \Delta^2$.

20. i) $\sum a^3\sin(B-C) = 0$.

ii) $\sum a^3\cos(B-C) = 3abc$

iii) $a^2\sin 2B + b^2\sin 2A = 4\Delta$

21. i) $\cot A + \cot B + \cot C = \dfrac{a^2+b^2+c^2}{4\Delta}$

ii) $\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2} = \dfrac{(a+b+c)^2}{4\Delta}$.

22. i) If $a\cos B = b\cos A$, then the triangle is isosceles.

ii) If $a\cos A = b\cos B$, then the triangle is isosceles or right angled.

iii) If $a^2 + b^2 + c^2 = 8R^2$, then the triangle is right angled.

iv) If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then the triangle is right angled.

v) If $\cos A = \dfrac{\sin B}{2\sin C}$, then the triangle is isosceles.

vi) If $\dfrac{a}{\cos A} = \dfrac{b}{\cos B} = \dfrac{c}{\cos C}$, then the triangle is equilateral.

vii) If $\cos A + \cos B + \cos C = 3/2$, then the triangle is equilateral.

viii) If $\sin A + \sin B + \sin C = \dfrac{3\sqrt{3}}{2}$, then the triangle is equilateral.

ix) If $\cot A + \cot B + \cot C = \sqrt{3}$, then the triangle is equilateral.

23. i) If $\dfrac{a^2+b^2}{a^2-b^2} = \dfrac{\sin(A+B)}{\sin(A-B)}$, then $C = 90°$.

ii) If $\dfrac{a+b}{b+c} + \dfrac{b}{c+a} = 1$, then $C = 60°$.

iii) If $\dfrac{1}{a+b} + \dfrac{1}{a+c} = \dfrac{3}{a+b+c}$, then $A = 60°$

iv) If $\dfrac{b}{a^2-c^2} + \dfrac{c}{a^2-b^2} = 0$, then $A = 60°$.

i) a, b, c are In H.P. $\Leftrightarrow \sin^2\dfrac{A}{2}, \sin^2\dfrac{B}{2}, \sin^2\dfrac{C}{2}$ are in H.P.

ii) a, b, c are in A.P. $\Leftrightarrow \cot\dfrac{A}{2}, \cot\dfrac{B}{2}, \cot\dfrac{C}{2}$ are in A.P.

iii) a, b, c are in A.P. $\Leftrightarrow \tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ are in H.P.

iv) a^2, b^2, c^2 are in A.P. $\Leftrightarrow$ cotA, cotB, cotC are in A.P.

v) a^2, b^2, c^2 are in A.P. $\Leftrightarrow$ tanA, tanB, tanC are in H.P